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THE RECONSTRUCTION OF CONTROLS IN NON-LINEAR DISTRIBUTED SYSTEMS†

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The dynamical reconstruction of unknown distributed and boundary controls applied to non-linear equations of parabolic and hyperbolic type is discussed. Regularizing algorithms are indicated that enable the controls to be reconstructed synchronously with the evolution of the processes in question. The algorithms are stable with respect to information noise and computation errors.

Problems of the dynamical reconstruction of the unknown inputs to systems with distributed parameters are discussed. These inputs may be distributed and boundary controls occurring in Dirichlet boundary-value problems. It is assumed that the system, governed by a parabolic or hyperbolic equation, functions over a given time interval $T = [0, \vartheta]$. The evolution of its phase state $z(t)$, $t \in T$ (the trajectory of the system), is determined by a certain input (control) $u(\cdot)$ which may belong to a given function set $P(\cdot)$. The input $u(\cdot)$ itself and the phase trajectory $z(\cdot)$ of the system are unknown. However, certain devices are available that enable one, at discrete and fairly frequent times $\tau_i \in T$, $\tau_i < \tau_{i+1}$, to measure the error in the output $z(\tau_i)$. It is required to reconstruct the control $u_*(\cdot)$ that generates $z(\cdot)$: $u_*(\cdot) = u_*(\cdot; z(\cdot))$. Since exact reconstruction of $u_*(\cdot)$ is impossible, one has to devise an algorithm which computes some approximation of $u_*(\cdot)$. This approximation should be better the smaller the error in measuring $z(\tau_i)$ and the finer the partition $\{\tau_i\}$ of the interval T.

The problems discussed belong to the class of inverse problems of the dynamics of controllable systems (reconstruction of the input based on measurements of the output). Inverse problems for equations with distributed parameters have been investigated in an *a posteriori* setting by many authors [1–3]. An algorithm has been proposed for the dynamical reconstruction of the input to a finite-dimensional dynamical system which is affine with respect to the control [4]. This method, which can be used effectively [5, 6] to solve various inverse problems for systems described by ordinary differential equations, is based on ideas of positional control theory [7] and the methods of smoothing functionals and residuals familiar from the theory of ill-posed problems [1]. The method has been further developed for various classes of systems with distributed parameters [8–11]. These studies have discussed problems of the dynamical reconstruction of distributed and boundary controls, as well as the coefficients of an elliptical operator.

The aim of this paper is, relying on boundary control theory as presented in [12–16], to demonstrate the possibilities of the method of auxiliary positional-control models for investigating problems of reconstructing unknown distributed and boundary controls applied to non-linear equations of parabolic and hyperbolic types. In the investigation of parabolic systems, use is made of the technique of contracting semi-groups. Hyperbolic objects are considered using the "cosine" operator (concerning this approach, see [12, 15, 16]).

1. THE RECONSTRUCTION OF CONTROL IN PARABOLIC SYSTEMS

Let Σ be a controllable system described by a parabolic equation

$$x_t(t,\eta) - \Delta x(t,\eta) = f(t,\eta) + (B_1 u_1(t))(\eta) + \Phi(x(t,\eta)) \quad (1.1)$$

in $T \times \Omega = Q$, with initial conditions

$$x(0,\eta) = x_0(\eta) \text{ in } \Omega \quad (1.2)$$

and boundary conditions

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$$x(t)|_{\Gamma} = B_2 u_2(t), \quad t \in T \tag{1.3}$$

where $\Omega \subset R^n$ is an open bounded domain with a sufficiently smooth boundary Γ, Δ is the Laplacian, $f(\cdot) \in L_2(T; L_2(\Omega))$ is a given perturbation, $\Phi(\cdot)$ is a function satisfying the Lipschitz condition, and $B_1 \in L(U_1; L_2(\Omega))$ and $B_2 \in L(U_2; L_2(\Gamma))$ are continuous linear operators, U_1 and U_2 being uniformly convex Banach spaces.

Following the approach used in [13], we will define what is meant by a solution. Let σ be a Dirichlet operator, that is

$$\sigma u_2 = h \Leftrightarrow \begin{cases} \Delta h = 0 & \text{on } \Omega \\ h = u_2 & \text{on } \Gamma, \quad u_2 \in L_2(\Gamma) \end{cases}$$

It is well known [14, 17] that the operator σ is continuous from the space $L_2(\Omega)$ to the space H. We define a mapping

$$t \to p(t;\cdot,\cdot,\cdot) : H \times L_2(T;U) \times C(T;H) \to C(T;H)$$

$$p(t; x_0, u(\cdot), z(\cdot)) = S(t)x_0 + A\int_0^t S(t-\tau)\sigma B_2 u_2(\tau)d\tau +$$

$$+\int_0^t S(t-\tau)\{f(\tau) + B_1 u_1(\tau) + \Phi(z(\tau))\}d\tau, \quad t \in T$$

where

$$Ax = \Delta x, \quad x \in D(A) = H_1^0(\Omega) \cap H_2(\Omega)$$

is a generator of a contracting semigroup of linear continuous operators $\{S(t); t \geqslant 0\}$ on H, and $D(A)$ is the domain of definition of the operator A. By a solution of problem (1.1)–(1.3) corresponding to a control $u(\cdot) \in P(\cdot)$ we mean the unique function

$$x(\cdot) = x(\cdot; 0; x_0, u(\cdot)) \in C(T; H)$$

satisfying the integral equation

$$x(t) = p(t; x_0, u(\cdot), x(\cdot)), \quad t \in T$$

We will now formulate the problem considered in this section. System (1.1)–(1.3) receives unknown inputs $u_1(\cdot)$ and $u_2(\cdot)$, $u(t) = \{u_1(t), u_2(t)\} \in P = P_1 \times P_2$ for almost every (a.e.) $t \in T$; $P_1 \subset U_1$, $P_2 \subset U_2$ are convex, bounded and closed sets. At sufficiently frequent discrete times

$$\tau_i \in T, \quad \tau_i = \tau_{i-1} + \delta, \quad i \in [1 : m-1], \quad \tau_0 = 0, \quad \tau_m = \vartheta$$

the phase states $z(\tau_i) = x(\tau_i, \eta) = x(\tau_i; 0, x_0, u(\cdot)) \in H = L_2(\Omega)$. of system (1.1)–(1.3) are measured (with an error). The measurement results $\xi_i \in H$ satisfy the inequalities

$$|\xi_i - x(\tau_i)|_H \leqslant h \tag{1.4}$$

where h is the measurement accuracy parameter.

It is required to indicate an algorithm for reconstructing an unknown input

$$u_*(\cdot) = \{u_1^*(\cdot), \; u_2^*(\cdot)\} \in P(\cdot) = \{u(\cdot) = \{u_1(\cdot), \; u_2(\cdot)\} \in L_2(T; U):$$

$$u_1(t) \in P_1, \quad u_2(t) \in P_2 \quad \text{for a.e. } t \in T\}$$

which generates an unknown output $z(\cdot) = x(\cdot)$, that is, to construct an algorithm for the approximate computation of a control $u_*(\cdot)$ such that the corresponding solution $x(\cdot; 0, x_0, u_*(\cdot))$ is identical with $x(\cdot)$. Here $U = U_1 \times U_2$ is the space of controls, and $x(\cdot; 0, x_0, u_*(\cdot))$ is a solution of Eq. (1.1) satisfying initial conditions (1.2) and boundary conditions (1.3) and with the control $u(\cdot) = u_*(\cdot)$.

We will solve the problem along the lines of the approach proposed in [4–11]. It is first necessary to select an auxiliary system or model. As a model we take the linear system described by the parabolic equation

$$w_t(t,\eta)-\Delta w(t,\eta)=f(t,\eta)+(B_1 v_1^h(t))(\eta)+v_3^h(t,\eta) \text{ in } T\times\Omega \tag{1.5}$$

with initial condition

$$w(0,\eta)=w_0(\eta) \text{ in } \Omega \tag{1.6}$$

and the Dirichlet boundary condition

$$w(t)|_\Gamma = B_2 v_2^h(t), \quad t\in T \tag{1.7}$$

By a solution of Eq. (1.5) with initial condition (1.6) generated by controls $\{v_1^h(\cdot),\ v_2^h(\cdot)\}\in P(\cdot)$ and $v_3^h(\cdot)\in L_2(T;H)$, we mean a function

$$w^h(\cdot)=w(\cdot;0,w_0,\ v^h(\cdot))\in C(T;H),\quad v^h(\cdot)=\{v_1^h(\cdot),\ v_2^h(\cdot),\ v_3^h(\cdot)\}$$

defined by the following equality [14]

$$w^h(t)=S(t)w_0+A\int_0^t S(t-\tau)\sigma B_2 v_2^h(\tau)d\tau+\int_0^t S(t-\tau)\{f(\tau)+B_1 v_1^h(\tau)+v_3^h(\tau)\}d\tau,\quad t\in T$$

As is well known [14], if the condition

$$v^h(\cdot)\in P(\cdot)\times L_\infty(T;H) \tag{1.8}$$

is satisfied, such a solution exists and is unique.

Note that if $u(\cdot)$ (or $v^h(\cdot)$) is an unbounded function of time (e.g. a function which is summable in the norm squared), then the solution of Eq. (1.1) (or (1.5)) need not be an element of the space $C(T;H)$.

Let $\varphi(\cdot)$ be the modulus of continuity of the function $t\to x(t) = x(t;0,x_0,u(\cdot))\in H$ on T, that is

$$\varphi_x(\delta)=\sup\{|x(t_1)-x(t_2)|_H : t_1,\ t_2\in T,\ |t_1-t_2|<\delta\} \tag{1.9}$$

Let $\{\Delta_h\}$ be a family of uniform partitions

$$\Delta_h=\{\tau_i\}_{i=0}^m,\quad \tau_i=\tau_{h,i},\quad m=m_h,\quad \tau_0=0,\quad \tau_m=\vartheta \tag{1.10}$$

of the interval T with diameters $\delta = \delta(h)$.

We will now describe an algorithm for solving the problem. First we choose a family $\{\Delta_h\}$ and a function $\alpha(h)$: $[0, 1)\to R^+$ satisfying the following parameter-compatibility condition.

Condition 1.

$$\delta(h)\to 0+,\ \alpha(h)\to 0+$$

$$\{\delta(h)+h+\varphi_x(\delta(h))\}\alpha^{-1}(h)\to 0 \text{ as } h\to 0$$

Once h is entered (before the beginning of the process), it is fixed. Also fixed thereby are Δ_h and $\alpha(h)$. The operation of the algorithm is divided into $m_h - 1$ steps of the same type. During the ith step, implemented in the time interval $\delta_i = \delta_{h,i} = \{\tau_i, \tau_{i+1}), \tau_i = \tau_{h,i}$, the following operations are performed First, at time τ_i one finds a control

$$v^h(t)=\{v_1^h(t),\quad v_2^h(t),\quad v_3^h(t): t\in\delta_i\}$$

where

$$v_1^h(t)=v_{1i}^h,\ v_2^h(t)=v_*(t-\tau_i),\ v_3^h(t)=\Phi(\xi_i) \text{ for a.e. } t\in\delta_i$$

$$v_1^h=\arg\min\{2(s_i^*,A^{-1}B_1 v_1)_H+\alpha(h)|v_1|_{U_1}^2 : v_1\in P_1\}$$

$$v_*(\cdot)=\arg\min\left\{\int_{\tau_i}^{\tau_{i+1}}\{2(\sigma^* S(\tau_{i+1}-s)s_i^*, B_2 v_2(s))_{L_2(\Gamma)}+\right. \tag{1.11}$$

$$+\alpha(h)|v_2(s)|^2_{U_2}\}ds : v_2(s) \in P_2 \text{ for a.e. } s \in [0, \delta]\}$$

$$s_i^* = A^{-1}(\psi_i - \xi_i), \quad |\psi_i - w^h(\tau_i)|_H \le h$$

where the operator σ^* is the adjoint of the Dirichlet operator σ. This control is now applied at the input of the model. After that, the phase state of the model is recalculated: instead of $w(\tau_i)$ one finds $w(\tau_{i+1})$. The whole procedure is implemented before the time ϑ.

Remark. As is well known [16, 17]

$$\sigma^* z = \partial f / \partial \eta|_\Gamma, \quad \forall z \in H$$

where f is a solution of the Dirichlet problem

$$\Delta f(\eta) = z(\eta), \quad \eta \in \Omega; \quad f|_\Gamma = 0$$

Therefore

$$v_*(\cdot) = \arg\min\left\{\int_{\tau_i}^{\tau_{i+1}} \{2\left(\frac{\partial}{\partial \eta}\Delta^{-1}S(\tau_{i+1} - s)s_i^*|_\Gamma, \ B_2 v_2(s)\right)_{L_2(\Gamma)} + \right.$$

$$\left. +\alpha(h)|v_2(s)|^2_{U_2}\}ds : v_2(s) \in P_2 \text{ for a.e. } s \in [0, \delta]\right\}$$

Let $U(x(\cdot))$ be the set of all controls in $P(\cdot)$ compatible with the output $x(\cdot)$, that is

$$U(x(\cdot)) = \{u(\cdot) = \{u_1(\cdot), u_2(\cdot)\} \in P(\cdot) :$$

$$x(t) - S(t)x_0 - \int_0^t S(t-\tau)\{f(\tau) + \Phi(x(\tau))\}d\tau =$$

$$\left. = A\int_0^t S(t-\tau)\sigma B_2 u_2(\tau)d\tau + \int_0^t S(t-\tau)B_1 u_1(\tau)d\tau, \ \forall t \in T\right\} \tag{1.12}$$

This set is obviously convex, bounded and closed in $L_2(T; U)$. Hence it contains a unique element $u_*(\cdot) = u_*(\cdot; x(\cdot)) = \{u_{1*}(\cdot), u_{2*}(\cdot)\}$ of minimum $L_2(T; U)$-norm.

Let $v^h_* = \{v_1^h(\cdot), v_2^h(\cdot)\}$.

Theorem 1. Suppose Condition 1 is satisfied and the initial state $w_0 \in H$ of the model is such that

$$|x_0 - w_0|_H \le h$$

Then, for any $\varepsilon > 0$, one can find an h_1 such that, whenever $h \le h_1$

$$|v_*^h(\cdot) - u_*(\cdot; x(\cdot))|_{L_2(T;U)} \le \varepsilon \tag{1.13}$$

We define the Lyapunov functional

$$\varepsilon(t) = |A^{-1}(w(t) - x(t))|^2_H + E(t) \tag{1.14}$$

$$E(t) = \alpha(h)\int_0^t \left\{|v_2^h(s)|^2_{U_2} + |v_1^h(s)|^2_{U_1} - |u_{2*}(s)|^2_{U_2} - |u_{1*}(s)|^2_{U_1}\right\}ds \tag{1.15}$$

Lemma 1. The following inequality holds uniformly over all partitions Δ_h and measurements ξ_i that satisfy inequalities (1.4)

$$\varepsilon(\tau_i) \le k_1(\delta + h + \varphi_x(\delta)), \quad i = 1, \dots, m$$

The proof of Theorem 1 is based on this lemma, whose truth, in turn, is established in the same way as in [4, 9].

2. RECONSTRUCTION OF THE CONTROL IN HYPERBOLIC SYSTEMS

Let Σ be a controllable system described by a hyperbolic equation

$$x_{tt}(t,\eta)-\Delta x(t,\eta)=f(t,\eta)+(B_1u_1(t))(\eta)+ +\Phi(x(t,\eta),x_t(t,\eta)) \text{ in } T\times\Omega=Q \tag{2.1}$$

with initial conditions

$$x(0,\eta)=x_0(\eta),\ \ x_t(0,\eta)=x_{10}(\eta) \text{ in } \Omega \tag{2.2}$$

and boundary conditions (1.3). The domain Ω, the operator Δ, the function $f(\cdot)$, the mappings B_1, B_2 and the spaces U_1, U_2, U, V are the same as in Section 1. The non-linear perturbation $\Phi(\cdot,\cdot)$ is Fréchet differentiable on $H\times H^{-1}(\Omega)$ ($H=L_2(\Omega)$) and satisfies the growth condition

$$|\Phi'(x,y)|_H\leqslant M\{|x|_H+|y|_{H^{-1}(\Omega)}\},\quad \forall(x,y)\in H\times H^{-1}(\Omega)$$

(Φ' denotes the Fréchet derivative).

Following the approach described in [13], we define what is meant by a solution of problem (2.1), (2.2), (1.3). Let $x_0\in H, x_{10}\in H^{-1}(\Omega)$. Define a mapping

$$t\to p(t;\cdot,\cdot,\cdot,\cdot)=\{p_1(\cdot),p_2(\cdot),\}:H\times H^{-1}(\Omega)\times L_2(T;U)\times \times C(T;H\times H^{-1}(\Omega))\to C(T;H\times H^{-1}(\Omega))$$

$$p_j(t;x_0,x_{10},u(\cdot),z(\cdot))=S_j^0(t)x_0+S_j(t)x_{10}-A\int_0^t S_j(t-\tau)\sigma B_2u_2(\tau)d\tau+ +\int_0^t S_j(t-\tau)\{f(\tau)+B_1u_1(\tau)+\Phi(z_1(\tau),z_2(\tau))\}d\tau,\quad t\in T;\quad j=1,2$$

$$S_1^0(t)=C(t),\ \ S_2^0(t)=AT(t),\ \ S_1(t)=T(t),\ \ S_2(t)=C(t)$$

$$z(\cdot)=\{z_1(\cdot),z_2(\cdot)\}\in C(T;H\times H^{-1}(\Omega))$$

The operator A was defined in Section 1. It is well known [12] that this operator is a generator of a strongly continuous "cosine" operator $\{C(t);t\geqslant 0\}$ on H, with which, in turn, we can associate a "sine" operator

$$T(t)x=\int_0^t C(\tau)xd\tau$$

By a solution of problem (2.1), (2.2), (1.3) corresponding to a control $u(\cdot)\in P(\cdot)$ we mean, following the approach of [13, 15], the unique function

$$z(\cdot)=\{x(\cdot\,;0,x_0,x_{10},u(\cdot)),\ \ x_t(\cdot\,;0,x_0,x_{10},u(\cdot))\}\in C(T;H\times H^{-1}(\Omega))$$

satisfying the integral equation

$$z(t)=p(t;x_0,x_{10},u(\cdot),z(\cdot)),\quad t\in T$$

Essentially, our problem is analogous to that described in Section 1 for a parabolic equation. Namely, it is assumed that the system receives unknown inputs $u_1(\cdot)$ and $u_2(\cdot)$, $u(t)=\{u_1(t),u_2(t)\}\in P=P_1\times P_2$ for almost all $t\in T$; $P_1\subset U_1$ and $P_2\subset U_2$ are convex, bounded and closed sets. At sufficiently frequent discrete times

$$\tau_i\in T,\ \ \tau_i=\tau_{i-1}+\delta,\ \ i\in[1:m-1],\ \ \tau_0=0,\ \ \tau_m=\vartheta$$

the phase states

$$z(\tau_i) = \{x(\tau_i, \eta), x_t(\tau_i, \eta)\}$$

of system (2.1), (2.2), (1.3) are measured (with an error). The measurement results $\xi_i = \{\xi^{(1)}(\tau_i)\} \in H \times H^{-1}(\Omega)$ satisfy the inequalities

$$|\xi_i^{(1)} - x(\tau_i)|_H \leq h, \quad |\xi_i^{(2)} - x_t(\tau_i)|_{H^{-1}(\Omega)} \leq h \tag{2.3}$$

where h is the measurement accuracy parameter. The problem is to construct an algorithm for the dynamical reconstruction of the unknown input $u_*(\cdot) = \{u_1^*(\cdot), u_2^*(\cdot)\} \in P(\cdot)$ which also generates the unknown output $z(\cdot)$.

We will now solve the problem. As a model we take the linear system described by the hyperbolic equation

$$\begin{aligned} &w_{tt}(t,\eta) - \Delta w(t,\eta) = \\ &= f(t,\eta) + (B_1 v_1^h(t))(\eta) + v_3^h(t,\eta) \text{ in } T \times \Omega \end{aligned} \tag{2.4}$$

with initial conditions

$$w(0,\eta) = w_0(\eta), \quad w_t(0,\eta) = w_{10}(\eta) \text{ in } \Omega \tag{2.5}$$

and Dirichlet boundary conditions (1.7).

By a solution of Eq. (2.4) with initial conditions (2.5) generated by the controls $\{v_1^h(\cdot), v_2^h(\cdot)\} \in P$ and $v_3^h(\cdot) \in L_2(T; H)$ we mean a function

$$w^h(\cdot) = w(\cdot\,; 0, w_0, w_{10}, v^h(\cdot)) \in C(T; H), \quad w_t^h(\cdot) \in C(T; H^{-1}(\Omega))$$

$$v^h(\cdot) = \{v_1^h(\cdot), \quad v_2^h(\cdot), \quad v_3^h(\cdot)\}$$

defined by the following equalities [15]

$$w^h(t) = C(t)w_0 + T(t)w_{10} - A\int_0^t T(t-\tau)\sigma B_2 v_2^h(\tau)d\tau +$$

$$+ \int_0^t T(t-\tau)\{f(\tau) + B_1 v_1^h(\tau) + v_3^h(\tau)\}d\tau$$

$$w_t^h(t) = AT(t)x_0 + C(t)x_{10} - A\int_0^t C(t-\tau)\sigma B_2 u_2(\tau)d\tau +$$

$$+ \int_0^t C(t-\tau)\{f(\tau) + B_1 u_1(\tau) + v_3^h(\tau)\}d\tau, \quad t \in T$$

As is well known [15], if condition (1.8) is satisfied, such a solution exists and is unique.

Let $\varphi(\cdot)$ and $\varphi_{\dot{x}}(\cdot)$ be the moduli of continuity of the functions $t \to x(t) = x(t;\, 0;\, x_0, x_{10}, u_*(\cdot)) \in H$, $t \to x_t(t) = x_t(t;\, 0;\, x_0, x_{10}, u_*(\cdot)) \in H^{-1}(\Omega)$, respectively, on T, that is

$$\varphi_x(\delta) = \sup\{|x(t_1) - x(t_2)|_H:\ t_1, t_2 \in T,\ |t_1 - t_2| < \delta\}$$

$$\varphi_{\dot{x}}(\delta) = \sup\{|x_t(t_1) - x_t(t_2)|_{H^{-1}(\Omega)}:\ t_1, t_2 \in T,\ |t_1 - t_2| < \delta\}$$

Suppose, moreover, that we have chosen a family $\{\Delta_h\}$ of uniform partitions (1.10) of the interval T with diameters $\delta = \delta(h)$ and a function $\alpha(\cdot) : [0, 1) \to R^+$ satisfying the following parameter-compatibility condition.

Condition 2.

$$\delta(h) \to 0+, \quad \alpha(h) \to 0+$$

$$\{\delta(h) + h + \varphi_x(\delta(h)) + \varphi_{\dot{x}}(\delta(h))\}\alpha^{-1}(h) \to 0 \quad \text{as } h \to 0$$

The algorithm for solving the problem is analogous to that described in Section 1. One first selects a family $\{\Delta_h\}$ and function $\alpha(h)$ satisfying Condition 2. Once h is entered (before the beginning of the process), it is fixed. Also fixed thereby are Δ_h and $\alpha(h)$. The operation of the algorithm is divided into $m_h - 1$ steps of the same type. During the ith step, carried out in the time interval $\delta_i = \delta_{h,i} = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{h,i}$, the following operations are performed. First, at time τ_i one determines a control

$$v^h(t) = \{v_{1i}^h, v_{2i}^h, v_{3i}^h : t \in \delta_i\}$$

where

$$v_{1i}^h = \arg\min\{2(A^{-1}s_*^{(i)}, A^{-1}B_1 v_1)_H + \alpha(h)|v_1|_{U_1}^2 : v_1 \in P_1\}$$

$$v_{2i}^h = \arg\min\{2(s_*^{(i)}, A^{-1}\sigma B_2 v_2)_H + \alpha(h)|v_2|_{U_2}^2 : v_2 \in P_2\}$$

$$v_{3i}^h = \Phi(\xi_i^{(1)})$$

$$s_*^{(i)} = A^{-1}(\psi_i - \xi_i^{(2)}), \quad |\psi_i - w_t^h(\tau_i)|_{H^{-1}(\Omega)} \leqslant h$$

This control is then applied at the input of the model. After that, the phase state of the model is recalculated: instead of $w(\tau_i)$ one finds $w(\tau_{i+1})$. The whole procedure is implemented before the time ϑ.

Let $U(z(\cdot))$ be the set of all controls in $P(\cdot)$ compatible with the output $z(\cdot)$, that is, which satisfies a relationship analogous to (1.12), with $S(t)x_0$ replaced by $C(t)x_0 + T(t)x_{10}$ and $S(t-\tau)$ replaced by $T(t-\tau)$. This set is also convex, bounded and close in $L_2(T; U)$. It therefore contains a unique element $u_*(\cdot) = u_*(\cdot; z(\cdot)) = \{u_{1*}(\cdot; z(\cdot)), u_{2*}(\cdot; z(\cdot))\}$ of minimum $L_2(T; U)$-norm.

Let $v_1^h(\cdot) = \{v_1^h(\cdot), v_2^h(\cdot)\}$.

Theorem 2. Suppose Condition 2 is satisfied and the initial state $w_0^h \in H$ and $w_{10} \in H^{-1}(\Omega)$ of the model is such that

$$|x_0 - w_0|_H \leqslant h, \quad |x_{10} - w_{10}|_{H^{-1}(\Omega)} \leqslant h$$

Then, for any $\varepsilon > 0$, one can find h_2 such that, whenever $h \leqslant h_2$, inequality (1.13) holds.

We define a Lyapunov functional

$$\lambda(t) = \lambda^{(1)}(t) + \lambda^{(2)}(t) + E(t)$$

where

$$\lambda^{(1)}(t) = |A^{-1}y^h(t)|_H^2, \quad \lambda^{(2)}(t) = |A^{-1}y_t^h(t)|_{H^{-1}(\Omega)}^2$$

$$y^h(t) = w(t) - x(t)$$

The functional $E(t)$ is defined by formula (1.15).

The proof of the theorem relies on the following lemma.

Lemma 2. The following inequality is true uniformly over all partitions Δ_h and measurements ξ_i satisfying inequalities (2.3)

$$\lambda(\tau_i) \leqslant k(\delta + h + \varphi_x(\delta) + \varphi_x(\delta)), \quad i = 1, \dots, m$$

In the proof of Lemma 2, use is made of properties of the "sine" and "cosine" operator [12, 18], and the following inequalities are established

$$\lambda^{(1)}(\tau_{i+1}) \leqslant \lambda^{(1)}(\tau_i) + 2\delta(A^{-1}y(\tau_i), A^{-1}y_t(\tau_i))_{L_2(\Omega)} + K_1\delta(h + \delta)$$

$$\lambda^{(2)}(\tau_{i+1}) \leqslant \lambda^{(2)}(\tau_i) - 2\delta(A^{-1}y(\tau_i), A^{-1}y_t(\tau_i))_{L_2(\Omega)} + K_2\delta(h + \delta) +$$

$$+ \int_{\tau_i}^{\tau_{i+1}} [2(A^{-1}s_*^{(i)}, A^{-1}B_1(v_{1i}^h - u_{1*}(s)))_H + \alpha(h)\{|v_{1i}^h|_{U_1}^2 - |u_{1*}(s)|_{U_1}^2\}]ds +$$

$$+ \int_{\tau_i}^{\tau_{i+1}} [2(s_*^{(i)}, A^{-1}\sigma B_2(v_2 - u_{2*}(s)))_H + \alpha(h)\{|v_{2i}^h|_{U_2}^2 - |u_{2*}(s)|_{U_2}^2\}]ds$$

Remarks. 1. The problem of reconstructing controls in a hyperbolic system may also be solved using the semigroup approach, since, as is well known, the operator

$$A_* = \begin{Vmatrix} 0 & I \\ A & 0 \end{Vmatrix}$$

generates a C_0-semigroup in the space $H \times H^{-1}(\Omega)$ such that the solution of problem (2.1), (2.2), (1.3) may be represented in the form

$$\begin{Vmatrix} x(t) \\ \dot{x}(t) \end{Vmatrix} = \exp(A_* t) \begin{Vmatrix} x_0 \\ x_1 \end{Vmatrix} - A_* \int_0^t \exp(A_*(t-s)) \begin{Vmatrix} 0 \\ \sigma B_2 u_2(s) \end{Vmatrix} ds +$$

$$+ \int_0^t \exp(A_*(t-s)) \begin{Vmatrix} 0 \\ \Phi(x(s), \dot{x}(s)) + B_1 u_1(s) \end{Vmatrix} ds$$

2. The algorithms described in this paper may be suitable (with appropriate adjustments) for solving the problem of reconstructing boundary controls in the Dirichlet equation, in the case when the dynamical system under consideration is described by other types of non-linear equations with distributed parameters.

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